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THE BUFFERNESS PHENOMENON IN DISTRIBUTED MECHANICAL SYSTEM†

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Self-excited oscillations of weakly non-linear systems with distributed parameters are investigated. From the unified standpoints of the Lyapunov–Poincaré perturbation method, the limit cycles are determined in a constructive manner and conditions are found for the existence and stability of "quasi-linear" self-excited oscillatory modes of behaviour for two classes of mechanical objects, namely, a model of the transverse vibrations of a rotating thin shaft of circular cross-section, taking into account small internal and non-linear external viscous friction, and a two-dimensional model of the linear vibrations of a string connected at its midpoint to a self-excited oscillating circuit (an oscillator) of the Van der Pol type. In both models a "bufferness phenomenon" is established: the systems may have several stable limit cycles, depending on the values of the parameters (the angular velocity of rotation of the shaft or tension forces in the string) and corresponding to different oscillatory modes of distributed systems.

A bufferness phenomenon occurs in a system of partial differential equations if, subject to a suitable choice of the parameters, one can guarantee the existence in the system of any fixed number of attractors of the same type – equilibrium states, cycles (i.e. stable time-periodic solutions), tori, etc.

Based on investigations carried out to date, we can state that the phenomenon is fairly universal and is observed in mathematical models relating to various realms of nature: ecology [1, 2], radio-physics [3–5], etc. In particular, a bufferness phenomenon in radio-physics was predicted by A. A. Vitt [3] as far back as the early 1930s, and the results of much later research [6–8] imply that it is characteristic for a wide class of so-called self-excited generators with a section of long line in the feedback loop.

The aim of this paper is to show that the bufferness phenomenon is realizable in mathematical models of mechanical systems with distributed parameters.

1. THE BUFFERNESS PHENOMENON IN THE PROBLEM OF THE TRANSVERSE VIBRATIONS OF A ROTATING SHAFT

Formulation of the problem. Consider a mechanical system consisting of a flexible shaft of constant cross-section, of length l, rotating at a constant angular velocity ω. To describe a mathematical model of this system, we introduce coordinates, directing the x axis along the axis of the shaft; the u_1, u_2 axes will be considered fixed in a plane perpendicular to the axis of rotation (Fig. 1). Let $u_1(t, x)$, $u_2(t, x)$ be the coordinates of the displacement at time t of an arbitrary point of the shaft axis relative to the equilibrium position $u_1 = u_2 = 0$. Then, assuming that the shaft is made of a viscoelastic material and that the plane cross-section hypothesis holds, we obtain the following system of equations for the functions $u_j(t,x)(j = 1, 2)$ [9]

$$M\frac{\partial^2 u}{\partial t^2} + EJ\frac{\partial^4 u}{\partial x^4} + \omega \varkappa_1 EJA_0\frac{\partial^4 u}{\partial x^4} + \frac{\partial}{\partial t}\left[\varkappa_1 EJ\frac{\partial^4 u}{\partial x^4} + \varkappa_2 u\right] = 0 \qquad (1.1)$$

$$u = \operatorname{col}(u_1,\ u_2),\quad A_0 = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}$$

where EJ is the flexural stiffness of the shaft, M is its mass per unit of length, and $\varkappa_1$, $\varkappa_2$ are the coefficients of internal and external viscous friction.

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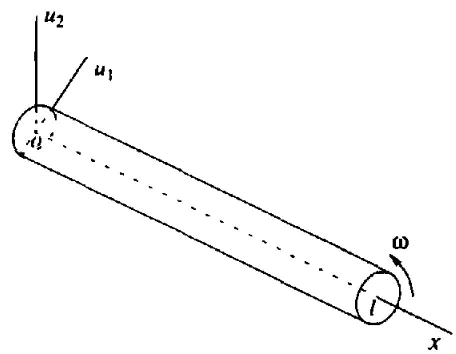

Fig. 1

We will assume that the ends of the shaft are suspended on hinges. Then the following boundary conditions must be added to system (1.1) in the interval $0 \leqslant x \leqslant l$:

$$u|_{x=0} = u|_{x=l} = \left.\frac{\partial^2 u}{\partial x^2}\right|_{x=0} = \left.\frac{\partial^2 u}{\partial x^2}\right|_{x=l} = 0 \tag{1.2}$$

We will assume that the coefficients M, EJ, and $\varkappa_1$ are constant, while the coefficient of external friction $\varkappa_2$ depends on $r = \sqrt{u_1^2 + u_2^2}$ as follows [9]:

$$\varkappa_2 = \alpha_1 + \alpha_2 r^2, \quad \alpha_j = \text{const} > 0, \quad j = 1, 2 \tag{1.3}$$

(by the symmetry of the problem, $\varkappa_2$ must indeed depend on r^2, and Eq. (1.3) is the simplest approximation of such a dependence in the neighbourhood of the point $r = 0$). Finally, keeping relations (1.3) in mind and successively normalizing

$$x/l \to x, \quad t\sqrt{EJ/(Ml^4)} \to t, \quad l^2\sqrt{\alpha_2/(\varkappa_1 EJ)}u \to u$$

we obtain the following equation instead of (1.1) (where I denotes the identity matrix)

$$\frac{\partial^2 u}{\partial t^2} + (I + \varepsilon\Omega A_0)\frac{\partial^4 u}{\partial x^4} + \varepsilon\frac{\partial^5 u}{\partial t \partial x^4} + \varepsilon\frac{\partial}{\partial t}[au + (u, u)u] = 0 \tag{1.4}$$

where

$$\varepsilon = \varkappa_1\sqrt{EJ/(Ml^4)}, \quad \Omega = \omega l^2\sqrt{M/(EJ)}, \quad a = \alpha_1 l^4/(\varkappa_1 EJ) \tag{1.5}$$

As the phase space (the space of the initial data $(u, \partial u/\partial t)$) of the boundary-value problem (1.2), (1.4) we take $\overset{\circ}{W}{}_2^4([0, 1]; \mathbb{R}^2) \times \overset{\circ}{W}{}_2^2([0, 1]; \mathbb{R}^2)$, where $\overset{\circ}{W}{}_2^4$, $\overset{\circ}{W}{}_2^2$ are the closures in the metrics of W_2^4 and W_2^2, respectively, of the linear space of smooth vector-functions satisfying boundary conditions (1.2) (by virtue of the normalization, we are now considering (1.2) with $l = 1$).

In order to prove that the mixed problem corresponding to problem (1.2), (1.4) has a locally unique solution in the phase space just indicated, we first put

$$h_1 = Bu, \quad h_2 = \frac{\partial u}{\partial t}, \quad Bu = -\frac{d^2 u}{dx^2}$$

As a result, we obtain the following system in the space $E = \overset{\circ}{W}{}_2^2([0, 1]; \mathbb{R}^2) \times \overset{\circ}{W}{}_2^2([0, 1]; \mathbb{R}^2)$

$$dh/dt = \mathcal{L}h + \varepsilon F(h) \tag{1.6}$$

where

$$h = \mathrm{col}(h_1, h_2), \quad \mathcal{L} = \Lambda_0 B + \varepsilon\Lambda_1 B + \varepsilon\Lambda_2 B^2 + \varepsilon a \Lambda_2$$

$$\Lambda_0 = \begin{Vmatrix} 0 & I \\ -I & 0 \end{Vmatrix}, \quad \Lambda_1 = \begin{Vmatrix} 0 & 0 \\ -\Omega A_0 & 0 \end{Vmatrix}, \quad \Lambda_2 = \begin{Vmatrix} 0 & 0 \\ 0 & -I \end{Vmatrix} \tag{1.7}$$

$$F(h) = \mathrm{col}(0, -(B^{-1}h_1, B^{-1}h_1)h_2 - 2(h_2, B^{-1}h_1)B^{-1}h_1)$$

The next stage of the proof involves a consideration of the closed linear operator $\mathcal{L}$ with a domain of definition dense in E. Relying on the Fourier method with respect to the system $\sin n\pi x (n = 1, 2, \ldots)$, it can be shown that the eigenvalues of the operator $\mathcal{L}$ are the roots of the equations

$$\lambda^2 + \varepsilon(a + \pi^4 n^4)\lambda + \pi^4 n^4 (1 - i\varepsilon\Omega) = 0, \quad n = 1, 2, \ldots \tag{1.8}$$

and of the equations obtained from (1.8) by complex conjugation. Note that the roots of these equations fall into two sequences $\lambda_n^1, \bar{\lambda}_n^1$ and $\lambda_n^2, \bar{\lambda}_n^2$ $(n = 1, 2, \ldots)$ such that

$$\mathrm{Re}\,\lambda_n^1 \to -\infty, \quad \lambda_n^2 \to -(1 - i\varepsilon\Omega)/\varepsilon, \quad n \to \infty$$

Hence it follows that the operator $\mathcal{L}$ generates in E an analytic semigroup $T(t), t \geqslant 0$, of linear bounded operators strongly continuous at the point $t = 0$.

In the concluding stage of the proof we consider system (1.6) together with an arbitrary initial condition $h|_{t=0} = h_0 \in E$ and transfer, using the semigroup $T(t)$, from this initial-value problem to an integral equation

$$h(t) = T(t)h_0 + \varepsilon \int_0^t T(t - \tau) F(h(\tau)) d\tau \tag{1.9}$$

Taking into account that the superposition operator $F(*) : E \to E$ is smooth in Frechet's sense, and applying the principle of contraction mapping to Eq. (1.9) in a certain sphere of the space $C([0, t_0]; E)$, where $t_0 > 0$ is suitably small, we obtain the required fact.

The reader's attention is drawn to the fact that if $t > 0$ the solutions we have constructed of problem (1.2), (1.4) are continuously differentiable with respect to t, as may be verified by applying the standard technique of "inflation" of smoothness to integral equation (1.9) (see, e.g. [10]). The smoothness of these solutions with respect to x is not changed, remaining the same as at $t = 0$.

Let us investigate the stability of the trivial equilibrium position of problem (1.2), (1.4). Analysis of Eqs (1.8) shows that it is stable provided that

$$\Omega < \min_{n \geqslant 1} \Omega_n, \quad \Omega_n = \omega_n + a/\omega_n, \quad \omega_n = n^2\pi^2 \tag{1.10}$$

As the parameter Ω is increased, each time it passes through the critical values $\Omega_n, n \geqslant 1$, exactly one complex root $\lambda = \lambda_n(\Omega, \varepsilon), \lambda_n(\Omega_n, \varepsilon) \equiv i\omega_n$ of Eq. (1.8) passes from the left complex half-plane into the right half-plane and remains there for all $\Omega > \Omega_n$.

The problem that immediately comes to mind is whether t-periodic solutions of problem (1.2), (1.4) exist that bifurcate from zero as the angular velocity of rotation Ω is increased and passes through its critical values $\Omega_n, n \geqslant 1$, and whether these solutions are stable. This problem will now be investigated in the case when $\varepsilon \ll 1$ and the parameters Ω and a are of the order of unity.

The existence of self-similar cycles. Putting $\xi = u_1 + iu_2$, we rewrite the boundary-value problem (1.2), (1.4) in complex form

$$\frac{\partial^2\xi}{\partial t^2} + (1 - i\varepsilon\Omega)\frac{\partial^4\xi}{\partial x^4} + \varepsilon\frac{\partial^5\xi}{\partial t\partial x^4} + \varepsilon\frac{\partial}{\partial t}[a\xi + |\xi|^2\xi] = 0$$

$$\xi|_{x=0} = \xi|_{x=1} = \left.\frac{\partial^2\xi}{\partial x^2}\right|_{x=0} = \left.\frac{\partial^2\xi}{\partial x^2}\right|_{x=1} = 0 \tag{1.11}$$

assuming that $\bar{\xi}$ satisfies the complex-conjugate boundary-value problem. A self-similar cycle of the problem thus obtained is a periodic solution of the form

$$\xi(t,\ x,\ \varepsilon)=\xi(x,\ \varepsilon)\exp(i\psi(\varepsilon)t) \tag{1.12}$$

The problem of the existence of such a solution reduces to finding the complex amplitude $\xi(x, \varepsilon)$ and real constant ψ from the non-linear boundary-value problem

$$(1+i\varepsilon(\psi-\Omega))\frac{d^4\xi}{dx^4}-\psi^2\xi+i\varepsilon\psi(a+|\xi|^2)\xi=0$$
$$\xi|_{x=0}=\xi|_{x=1}=\left.\frac{d^2\xi}{dx^2}\right|_{x=0}=\left.\frac{d^2\xi}{dx^2}\right|_{x=1}=0 \tag{1.13}$$

Now, then $\varepsilon = 0$ problem (1.11) admits of trigonometric solutions

$$\xi=\exp(\pm i\omega_n t)\sin n\pi x,\quad n\geqslant 1$$

Hence, if $\varepsilon > 0$ is small, it is natural to look for cycles (1.12) with frequencies ψ close to $\pm\omega_n$, $n \geqslant 1$.

We will first consider an algorithm for constructing the asymptotic form of self-similar cycles (1.12). To that end, we fix an arbitrary natural number n and consider (1.13) with

$$\psi=\omega_n\sigma(\varepsilon),\quad \sigma(\varepsilon)=1+\varepsilon\sigma_1+\varepsilon^2\sigma_2+\dots$$
$$\xi(x,\ \varepsilon)=\eta_0\sin n\pi x+\varepsilon\xi_1(x)+\varepsilon^2\xi_2(x)+\dots \tag{1.14}$$

where η_0, σ_1, σ_2, etc. are unknown real constants. Equating the coefficients of ε, we obtain the boundary-value problem

$$L\xi_1=f_1(x),\quad \xi_1|_{x=0,\,x=1}=\left.\frac{d^2\xi_1}{dx^2}\right|_{x=0,\,x=1}=0 \tag{1.15}$$

where

$$L=\frac{d^4}{dx^4}-\omega_n^2,\quad f_1=[(\Omega-\omega_n)\omega_n-a-\eta_0^2\sin^2 n\pi x-2i\omega_n\sigma_1]i\omega_n\eta_0\sin n\pi x$$

The condition for this problem to be solvable, i.e. for the functions $f_1(x)$ and $\sin n\pi x$ to be orthogonal in the sense of $L_2(0, 1)$, leads in turn to the equality

$$\omega_n(\Omega-\Omega_n)\eta_0-2i\omega_n\sigma_1\eta_0-3\eta_0^3/4=0 \tag{1.16}$$

Let us assume that the condition for self-excitation of self-excited oscillations at frequency ω_n, i.e. the inequality

$$\Omega>\Omega_n \tag{1.17}$$

is satisfied (see (1.10)). Then we find from (1.16) that

$$\sigma_1=0,\quad \eta_0=\sqrt{4/3\,\omega_n(\Omega-\Omega_n)} \tag{1.18}$$

and then, using (1.15), we define the function

$$\xi_1(x)=\frac{i\eta_0^3}{320\omega_n}\sin 3n\pi x+\eta_1\sin n\pi x \tag{1.19}$$

where η_1 is an arbitrary real constant.

Note that this algorithm may be continued indefinitely: the solvability of the linear inhomogeneous boundary-value problems analogous to (1.15) for $\xi_j(x)$, $j \geqslant 1$ is obtained by correcting the frequency σ_j and the term $\eta_{j-1}\sin n\pi x$ to within which $\xi_{j-1}(x)$ is determined. When that is done, unlike bifurcation equation (1.16) for η_{j-1}, σ_j, $j \geqslant 2$, one obtains linear inhomogeneous equations of the form

$$-\tfrac{3}{2}\eta_0^2\eta_{j-1} - 2i\omega_n\eta_0\sigma_j = \varphi_j, \quad j \geqslant 2 \tag{1.20}$$

from which these constants are uniquely determined.

The construction just described is made rigorous by the following proposition.

Theorem 1. Let the parameter Ω be fixed and satisfy condition (1.17). Then, for all sufficiently small ε, boundary-value problem (1.11) has a self-similar cycle (1.12), whose amplitude $\xi(x, \varepsilon)$ and frequency $\psi(\varepsilon)$ are analytic functions of their arguments and satisfy the equalities

$$\xi(x, 0) = \eta_0 \sin n\pi x; \quad \psi(\varepsilon) = \omega_n\sigma(\varepsilon), \quad \sigma(0) = 1, \quad \sigma'(0) = 0 \tag{1.21}$$

the constant η_0 being defined by the second equality of (1.18).

Proof. Divide the equation of (1.13) by $1 + i\varepsilon(\psi - \Omega)$, and then take it together with its own complex conjugate and substitute

$$\xi = \eta_0 \sin n\pi x + \varepsilon\xi_1(x) + \varepsilon^2 h_1, \quad \bar{\xi} = \eta_0 \sin n\pi x + \varepsilon\bar{\xi}_1(x) + \varepsilon^2 h_2$$
$$\psi = \omega_n(1 + \varepsilon^2\delta) \tag{1.22}$$

where the constant δ and the constant η_1 occurring in the definition of function $\xi_1(x)$ (see (1.19)) are assumed to be arbitrary, into the resulting system of equations. This finally yields an equation

$$\Pi h = G(x, h, \delta, \eta_1, \varepsilon) \tag{1.23}$$

where

$$h = \mathrm{col}(h_1, h_2), \quad \Pi = \mathrm{col}(L, L), \quad G(x, h, \delta, \eta_1, 0) = \mathrm{col}(\Delta, \bar{\Delta}) \tag{1.24}$$

$$\Delta = \omega_n^2[i(\Omega - \omega_n)\xi_1(x) + (2\delta - (\Omega - \omega_n)^2)\eta_0 \sin n\pi x] -$$
$$-i\omega_n[a\xi_1(x) + (2\xi_1(x) + \bar{\xi}_1(x))\eta_0^2 \sin^2 n\pi x] +$$
$$+\omega_n(\Omega - \omega_n)(a + \eta_0^2 \sin^2 n\pi x)\eta_0 \sin n\pi x \tag{1.25}$$

(the operator L is the same as in problem (1.15)).

It can be seen that in the subspace $V_0 \subset \mathring{W}_2^4 \times \mathring{W}_2^4$ of vector-functions (with complex components) orthogonal in the sense of $L_2(0, 1) \times L_2(0, 1)$ to the vector-functions

$$e_1 \sin n\pi x, \quad e_2 \sin n\pi x; \quad e_1 = \mathrm{col}(1, 0), \quad e_2 = \mathrm{col}(0, 1) \tag{1.26}$$

the operator Π has a bounded inverse. Therefore, suitably correcting the inhomogeneity G and inverting Π, we change from Eq. (1.23) to an integral equation in the space V_0

$$h = \Pi^{-1}(G - \gamma_1 e_1 \sin n\pi x - \gamma_2 e_2 \sin n\pi x) \tag{1.27}$$

where

$$\gamma_j = 2\int_0^1 (G, e_j)\sin n\pi x dx, \quad j = 1, 2 \tag{1.28}$$

Note that the right-hand side of Eq. (1.27) generates in V_0 a non-linear operator F that is an analytic function both of h and of the parameters δ, η_1 and ε. In addition, it follows from (1.24) and (1.25) that the operator F maps some sphere about zero in the space V_0, of radius independent of ε, into itself and is a contraction operator in that sphere (with contraction constant of the order of ε). Hence, by the principle of contraction mapping (for a suitable version of this principle see, e.g., [11]), Eq. (1.27) uniquely defines a vector-function which is analytic in ε, δ and η_1 (in the metric of V_0)

$$h_0 = \mathrm{col}(h_1^0(x, \delta, \eta_1, \varepsilon), h_2^0(x, \delta, \eta_1, \varepsilon)) \tag{1.29}$$

In addition, it follows from the structure of Eqs (1.23) and (1.27) that

$$h_2^0 = \bar{h}_1^0 \tag{1.30}$$

The final stage of the proof involves determining the arbitrary real constants δ and η_1 from the condition that the corrections to the inhomogeneity G, occurring in (1.27), should vanish. Substituting vector-function (1.29) into (1.28), we obtain complex-valued functions $\gamma_j(\delta, \eta_1, \varepsilon)(j = 1, 2)$, analytic in all their variables, where, by virtue of the structure of G and equality (1.30), we have $\gamma_2 = \bar{\gamma}_1$. Hence the conditions $\gamma_1 = \gamma_2 = 0$ are equivalent to the requirement that

$$\gamma_1(\delta, \eta_1, \varepsilon) = 0 \tag{1.31}$$

To analyse Eq. (1.31), we again appeal to relations (1.24) and (1.25), from which it follows that when $\varepsilon = 0$ it converts into the following analogue of Eq. (1.20)

$$-\tfrac{3}{2}\eta_0^2\eta_1 - 2i\omega_n\eta_0\delta = ia\eta_0(\Omega - \omega_n)$$

which obviously has the unique solution

$$\eta_1 = 0, \quad \delta = \delta_*; \quad \delta_* = a(\omega_n - \Omega)/(2\omega_n)$$

Hence, in turn, we conclude that the Implicit Function Theorem is applicable to Eq. (1.31) at the point $\varepsilon = \eta_1 = 0$, $\delta = \delta_*$. Thus, condition (1.31) uniquely defines the following functions, which are analytic in the neighbourhood of $\varepsilon = 0$

$$\eta_1 = \eta_1(\varepsilon), \quad \delta = \delta(\varepsilon) : \eta_1(0) = 0, \quad \delta(0) = \delta_* \tag{1.32}$$

And finally, substituting expressions (1.29) and (1.32) into system (1.22), we obtain the required self-similar cycle of problem (1.11).

Periodic solutions of the form (1.12) have a readily understandable mechanical interpretation. Arbitrarily fixing a number $x_0 \in (0, 1)$, consider the point of the shaft axis in the section $x = x_0$. It follows from solution (1.12) that this point will move in the plane $x = x_0$ about its equilibrium position $u_1 = u_2 = 0$, describing a circle of radius $r = |\xi(x_0, \varepsilon)$, at constant angular velocity $\psi(\varepsilon)$ (Fig. 2). This motion is called *steady asynchronous precession* [9]. The term "asynchronous" here means that, by condition (1.17), the angular velocity Ω at which the shaft will rotate about its axis is different from the angular velocity $\psi(\varepsilon)$, $\psi(0) = \omega_n$ of the precessional motion.

Note that problem (1.11) certainly has no self-similar cycles (1.12) with frequencies $\psi(\varepsilon)$, $\psi(0) = -\omega_n$. This is obvious from a mechanical standpoint, since the rotation of the shaft about its axis and the precession cannot take place in the same direction. This is also mathematically obvious: replacing ω_n by $-\omega_n$ in (1.14), we obtain an equation for the amplitude η_0 which has no real solutions: $\eta_0^2 = -4\omega_n(\Omega + \Omega_n)/3$.

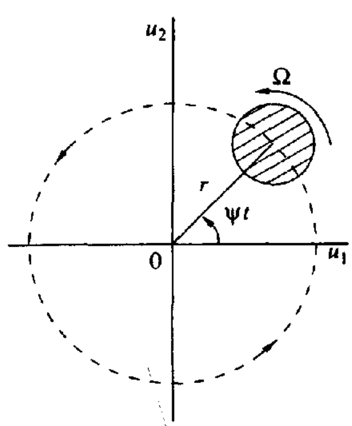

Fig. 2

The stability of self-similar cycles. To investigate the stability of the cycle (1.12), (1.21) for arbitrary fixed n, we consider the boundary-value problem

$$\lambda^2 v + \lambda \Pi_1 v + \Pi_2 v = 0,\quad v\,|_{x=0} = v\,|_{x=1} = \left.\frac{d^2 v}{dx^2}\right|_{x=0} = \left.\frac{d^2 v}{dx^2}\right|_{x=1} = 0 \tag{1.33}$$

where

$$v = \mathrm{col}(v_1,\ v_2),\quad \Pi_1 v = 2i\psi(\varepsilon)Q_0 v + \varepsilon\frac{d^4 v}{dx^4} + \varepsilon[Q_1(x,\varepsilon) + aI]v$$

$$\Pi_2 v = [I + i\varepsilon(\psi(\varepsilon) - \Omega)Q_0]\frac{d^4 v}{dx^4} - \psi^2(\varepsilon)v + i\varepsilon\psi(\varepsilon)Q_0[Q_1(x,\ \varepsilon) + aI]v$$

$$Q_0 = \mathrm{diag}\{1,\ -1\},\quad Q_1 = \begin{Vmatrix} p & q \\ \bar{q} & \bar{p} \end{Vmatrix},\quad p = 2|\xi(x,\ \varepsilon)|^2,\quad q = \xi^2(x,\ \varepsilon)$$

We emphasize that eigenvalue problem (1.33) is obtained from Eq. (1.11) and the complex conjugate of (1.11) by the substitutions

$$\xi\exp(-i\psi(\varepsilon)t) \to \xi,\quad \bar{\xi}\exp(i\psi(\varepsilon)t) \to \bar{\xi}$$

which transform the cycle (1.12) into an equilibrium state $(\xi(x,\,\varepsilon),\ \bar{\xi}(x,\,\varepsilon))$, followed by linearization at this equilibrium state. We also note that, by construction, problem (1.33) will always have the eigenvalue zero, to which the eigenfunction $\mathrm{col}(i\xi,\ -i\bar{\xi})$ corresponds. Finally, the reader's attention is drawn to the fact that if problem (1.33) has an eigenvalue λ_0 with eigenfunction $\mathrm{col}(v_1^0,\ v_2^0)$, then it will also have an eigenvalue $\bar{\lambda}_0$ with eigenfunction $\mathrm{col}(\bar{v}_2^0,\ \bar{v}_1^0)$.

When $\varepsilon = 0$ the spectrum of problem (1.33) consists of the group of eigenvalues $i(\omega_m - \omega_n)$, $i(\omega_m + \omega_n)$, $m \geqslant 1$, with eigenfunctions $e_1 \sin m\pi x$ and $e_2 \sin m\pi x$ respectively (the vectors e_1 and e_2 are the same as in (1.26)), together with the complex-conjugate eigenvalues. Therefore, an investigation of the stability of cycle (1.12), (1.21) in turn involves constructing the asymptotic form of these eigenvalues for small ε.

An algorithm to that end will first be described for the case of odd n. With $m \neq n$, $m = 1, 2, \ldots$, we substitute the following expressions into (1.33)

$$v = e_j \sin m\pi x + \varepsilon h_m^j(x) + \ldots,\quad \lambda = i(\omega_m + (-1)^j\omega_n) + \varepsilon\mu_m^j + \ldots,\quad j = 1,\ 2 \tag{1.34}$$

As a result, after equating the coefficients of ε, we obtain boundary-value problems

$$L_j h_m^j = \varphi_j(x),\quad h_m^j\,|_{x=0,\,x=1} = \left.\frac{d^2 h_m^j}{dx^2}\right|_{x=0,\,x=1} = 0,\quad j = 1,\ 2 \tag{1.35}$$

where

$$L_j = \frac{d^4}{dx^4} - H_j,\quad H_1 = \mathrm{diag}\{\omega_m^2,\ (\omega_m - 2\omega_n)^2\},\quad H_2 = \mathrm{diag}\{(\omega_m + 2\omega_n)^2,\ \omega_m^2\}$$

$$\varphi_j = -i\omega_m[2\mu_m^j + \omega_m(\Omega_m + (-1)^j\Omega)]e_j \sin m\pi x -$$

$$-i[(\omega_m + (-1)^j\omega_n)Q_1(x,\ 0) + \omega_n Q_0 Q_1(x,\ 0)]e_j \sin m\pi x$$

Considering the inhomogeneities $\varphi_j(x)$, we stipulate that the coefficients of $e_j \sin m\pi x$ should vanish; this yields

$$\mu_m^j = \tfrac{1}{2}[\omega_m((-1)^{j-1}\Omega - \Omega_m) - \tfrac{4}{3}\omega_n(\Omega - \Omega_n)],\quad j = 1,\ 2 \tag{1.36}$$

and the functions h_m^j may then be determined from problem (1.35) as linear combinations of harmonics $\sin m\pi x$, $\sin(m \pm 2n)\pi x$ with vector coefficients. We emphasize that this is possible by virtue of the inequalities

$$\omega_m^2 - (\omega_m \pm 2\omega_n)^2 \neq 0, \quad \omega_{m\pm 2n}^2 - (\omega_m + 2\omega_n)^2 \neq 0, \quad \omega_{m\pm 2n}^2 - (\omega_m - 2\omega_n)^2 \neq 0$$

which follow from the assumption that n is odd and from the condition $m \neq n$.

If $m = n, j = 2$, formulae (1.34) are unchanged, and the operations just described then yield the equality

$$\mu_n^2 = -\omega_n(3\Omega - \Omega_n)/2 \tag{1.37}$$

If $m = n, j = 1$, then in the first approximation, that is, when $\varepsilon = 0$, we have to deal with a zero eigenvalue of multiplicity 2; hence we have two linearly independent eigenfunctions $e_k \sin n\pi x$, $k = 1, 2$. Here, therefore, instead of (1.33), we consider the matrix boundary-value problem

$$V\Lambda^2 + \Pi_1 V\Lambda + \Pi_2 V = 0, \quad V|_{x=0,x=1} = \left.\frac{d^2V}{dx^2}\right|_{x=0,x=1} = 0 \tag{1.38}$$

where V and Λ are square matrices and the operators Π_1 and Π_2 are applied separately to each matrix column; for example, if $V = [v_1(x), v_2(x)]$, then $\Pi_2 V = [\Pi_2 v_1, \Pi_2 v_2]$. Further, setting

$$\Lambda = \varepsilon\Lambda_1 + \ldots, \; V = V_1(x) + \varepsilon V_2(x) + \ldots$$
$$V_1 = [e_1 \sin n\pi x, \quad e_2 \sin n\pi x] \tag{1.39}$$

in problem (1.38) and operating as described above, for the columns V_2 we obtain linear inhomogeneous boundary-value problems analogous to (1.35), and the elements of the constant matrix Λ_1 are determined from the conditions that these problems be solvable. Omitting the relatively simple calculations, we present the final result

$$\Lambda_1 = -\frac{\omega_n}{2}(\Omega - \Omega_n) \begin{Vmatrix} 1 & 1 \\ 1 & 1 \end{Vmatrix} \tag{1.40}$$

Note that one eigenvalue of the matrix (1.40) is zero (since 1.33) is linearization at a cycle), and the other is negative.

Now let us assume that n is even: $n = 2m_0$. It is obvious that if $m \neq m_0, 3m_0$, and also in the cases $m = m_0, j = 1$, and $m = 3m_0, j = 2$, all the constructions described above remain valid. The only cases requiring new attention are $m = 3m_0, j = 1$, and $m = m_0, j = 2$. Indeed, if $\varepsilon = 0$, we then have to deal with a double eigenvalue $\lambda = 5i\omega_{m0}$, to which correspond linearly independent eigenfunctions $e_1 \sin 3m_0\pi x$, $e_2 \sin m_0\pi x$. Therefore, to determine the asymptotic form of these eigenvalues, we substitute square matrices analogous to (1.39) into problem (1.38)

$$\Lambda = 5i\omega_{m_0} I + \varepsilon\Lambda_1 + \ldots, \quad V = V_1(x) + \varepsilon V_2(x) + \ldots \tag{1.41}$$
$$V_1 = [e_1 \sin 3m_0\pi x, \; e_2 \sin m_0\pi x], \quad V_2 = [v_{21}(x), \; v_{22}(x)]$$
$$\Lambda_1 = \|\lambda_{jk}\|, \quad j, \; k = 1, \; 2$$

and equate the coefficients of ε. As a result, we obtain the boundary-value problems

$$L_0 v_{2j} = \varphi_j(x), \quad v_{2j}|_{x=0,x=1} = \left.\frac{d^2 v_{2j}}{dx^2}\right|_{x=0,x=1} = 0, \quad j = 1, \; 2$$

where

$$L_0 = \frac{d^4}{dx^4} - \operatorname{diag}(\omega_{3m_0}^2, \; \omega_{m_0}^2)$$
$$\varphi_1 = -i(2\omega_{3m_0}\lambda_{11}e_1 \sin 3m_0\pi x + 2\omega_{m_0}\lambda_{21}e_2 \sin m_0\pi x) +$$
$$+ i\omega_{3m_0}^2(\Omega - \Omega_{3m_0})e_1 \sin 3m_0\pi x - i\omega_{m_0}(5I + 4Q_0)Q_1(x, \; 0)e_1 \sin 3m_0\pi x$$

$$\varphi_2 = -i(2\omega_{3m_0}\lambda_{12}e_1 \sin 3m_0\pi x + 2\omega_{m_0}\lambda_{22}e_2 \sin m_0\pi x) -$$

$$-i\omega_{m_0}^2(\Omega+\Omega_{m_0})e_2 \sin m_0\pi x - i\omega_{m_0}(5I+4Q_0)Q_1(x,\ 0)e_2 \sin m_0\pi x$$

The analysis of these problems is standard: first, considering the inhomogeneities φ_j ($j = 1, 2$) and equating the coefficients of $e_1 \sin 3m_0\pi x$, $e_2 \sin m_0\pi x$ to zero, we find the elements of Λ_1 and then also determine the functions v_{2j} ($j = 1, 2$). The corresponding calculations show that

$$\lambda_{11} = \mu_{3m_0}^1,\quad \lambda_{22} = \mu_{m_0}^2,\quad \lambda_{21} = \lambda_{12} = -\omega_{2m_0}(\Omega - \Omega_{2m_0})/6 \tag{1.42}$$

(for the definition of μ_m^j see (1.36) and (1.37)).

Thus, the stability of the cycle (1.12), (1.21) in the case of odd n depends on the signs of the numbers μ_m^1 with $m \geqslant 1, m \neq n$ determined in (1.36) (the numbers μ_m^2 need not be considered, since they are negative). In the case $n = 2m_0$, the stability of the cycle depends on the signs of the numbers μ_m^1 with $m \neq 3m_0, 2m_0$ and on the signs of the eigenvalues of the matrix with elements (1.42). Let us assume that none of these numbers vanishes and that m_* of them are positive. Then the following proposition is true.

Theorem 2. The cycle (1.12), (1.21) of boundary-value problem (1.11) is exponentially orbitally stable if $m_* = 0$ and dichotomic if $m_* > 0$, in which case the dimension of the unstable manifold is $2m_* + 1$.

Proof. Let $\lambda_m^j(\varepsilon)$ ($j = 1, 2$) denote the eigenvalues of problem (1.33) that become $i(\omega_m + (-1)^j\omega_n)$ when $\varepsilon = 0$. The validity of asymptotic equalities (1.34) for any fixed m, and of equalities (1.39) and (1.41), is readily established along the same lines as the analysis in non-linear problem (1.13). It should be emphasized, however, that these equalities are by no means uniformly applicable for all m. Therefore, in order to determine the asymptotic behaviour of $\lambda_m^j(\varepsilon)$ ($j = 1, 2$) as $m \to \infty$, one resorts to an auxiliary boundary-value problem, obtained from (1.33) by dropping the terms containing $Q_1(x, \varepsilon)$, which are not essential in this situation. The spectrum of this latter problem consists of the roots of the equations obtained from (1.8) by replacing n by m, and λ by $\lambda + i\psi(\varepsilon)$. Hence, in turn, we conclude that, first

$$\lambda_m^1(\varepsilon) \to -[1 + i\varepsilon(\psi(\varepsilon) - \Omega)]/\varepsilon,\quad \operatorname{Re}\lambda_m^2(\varepsilon) \to -\infty,\quad m \to \infty$$

and, second, we determine a sufficiently large natural number N, independent of ε, such that $\operatorname{Re}\lambda_m^j(\varepsilon) < 0$ ($j = 1, 2$) for all $m \geqslant N$.

Thus, the stability of the cycle (1.12), (1.21) depends only on a finite number of eigenvalues $\lambda_m^j(\varepsilon)$ ($j = 1, 2, m \leqslant N$). It is therefore determined by the signs of the numbers μ_m^1, and also by the eigenvalues of the matrix (1.42) in the case of odd n.

Conclusions. Let us analyse the conditions for stability of a cycle (1.12), (1.21) with fixed n in the case $a = 0$ that are most favourable for the generation of self-excited oscillations, since all conclusions obtained in that special case remain valid for $a > 0$. By the foregoing discussion, if $a = 0$ and n is odd, these conditions are

$$P(\sigma, m^2/n^2) < 0,\quad m = 1, 2, \ldots, m \neq n \tag{1.43}$$

$$P(\sigma, y) = \sigma y - y^2 - 4(\sigma - 1)/3,\quad \sigma = \Omega/\omega_n > 1$$

But if $n = 2m_0$, conditions (1.43) must hold for $m \neq 2m_0, 3m_0$ and, in addition, we have an additional condition, implying that the Hurwitz condition is satisfied for the matrix (1.42)

$$R(\sigma) \equiv \left(\frac{11}{12}\sigma - \frac{179}{48}\right)\left(-\frac{19}{12}\sigma + \frac{61}{48}\right) - \frac{1}{9}(\sigma - 1)^2 > 0 \tag{1.44}$$

Investigation of conditions (1.43) and (1.44) shows that for $n = 1, 2, 3$ the stable ranges of the cycles (1.12), (1.21) are given by the respective inequalities

$$\pi^2 < \Omega < 11\pi^2/2,\quad 61\pi^2/13 < \Omega < 4\pi^2\sigma_*,\quad 23\pi^2/2 < \Omega < 37\pi^2 \tag{1.45}$$

where $\sigma_* = 3.681 \ldots$ is the largest root of the equation $R(\sigma) = 0$. In particular, it follows from (1.45) that the stable cycles (1.12) and (1.21) with $n = 1, 2$ coexist in the interval (Ω_*, Ω_{**}), and the same is true of the stable cycles with $n = 2, 3$ in the interval $(\Omega_{***}, \Omega_{****})$, where

$$\Omega_* = 46.31\ldots, \quad \Omega_{**} = 54.28\ldots, \quad \Omega_{***} = 113.50\ldots, \quad \Omega_{****} = 145.32\ldots \tag{1.46}$$

(Fig. 3; the sections of the amplitude curves $\xi = \sqrt{4\omega_n(\Omega - \omega_n)/3}$ ($n = 1, 2, 3$) corresponding to stable cycles are shown by solid curves).

Thus, in the most interesting case $a = 0$, the stable periodic solutions of problem (1.11) depend on the parameter Ω as follows. At relative low angular velocities of rotation, namely, $\Omega < \pi^2$, the zero equilibrium position is stable (no precession). As Ω is increased and passes through its first critical value π^2, a stable cycle (1.12), (1.21) with $n = 1$ bifurcates from zero; it remains stable throughout the first interval (1.45) and becomes unstable when $\Omega > 11\pi^2/2$.

In the case of cycles (1.12) and (1.21) with $n \geq 2$, the situation is somewhat different. Each such cycle bifurcates from the zero equilibrium position as Ω passes through a critical value ω_n and initially is, of course, unstable. However, when Ω is increased further, it becomes stable, and will certainly remain stable in the interval

$$4\omega_n/3 < \Omega < 4\omega_n \text{ in the case of odd } n \tag{1.47}$$

$$4\omega_n/3 < \Omega < \sigma_*\omega_n \text{ in the case of even } n \tag{1.48}$$

Indeed, corresponding to the interval (1.47) we have $\sigma \in (4/3, 4)$, in which case, for any $y \geq 0$, we have the inequality $P(\sigma, y) < 0$. Thus all the stability conditions (1.43) are automatically satisfied in this interval. In the case of even n, the stable interval is shorter (compare conditions (1.47) and (1.48)), because $R(\sigma) > 0$ for $1 \leq \sigma < \sigma_*$ and $R(\sigma) < 0$ for $\sigma > \sigma_*$.

We might add that the actual stable ranges of the cycles (1.12) and (1.21) have the following properties. First, they are somewhat wider than intervals (1.47) and (1.48) (compare, for example, the second and third intervals (1.45) with intervals (1.48) for $n = 2$ and with (1.47) for $n = 3$). Second, these ranges are certainly finite, since $P(\sigma, m^2/n^2) \to \infty$ as $\sigma \to \infty$, for any fixed $m > 2n/\sqrt{3}$. Third, any two neighbouring ranges with indices n and $n + 1$ intersect: this may be verified directly when $n \leq 2$ (see Fig. 3), and when $n \geq 3$ it is a corollary of the fact that the corresponding intervals (1.47) and (1.48) intersect. Fourth, and finally, as Ω is increased, inequalities (1.47) and (1.48) begin to hold simultaneously for increasingly large numbers n, so that as $\Omega \to \infty$ and $\varepsilon \to 0$, the number of coexisting stable cycles (1.12), (1.21) of problem (1.11) increases without limit, that is, the *bufferness phenomenon* occurs.

In conclusion, we note that, since increasing Ω yields stable cycles (1.12), and (1.21) with increasingly high indices n, while cycles with low indices, conversely, lose their stability, it makes sense here to speak of a *high-mode bufferness property*. It should also be emphasized that, in principle, this increase in the number of stable cycles may be achieved only by increasing the shaft length l (with the other parameters left fixed and for sufficiently small a), since in that case one has a simultaneous increase in Ω and a decrease in ε (see 1.5)). However, in the most realistic parameter range (see (1.46)), we have one or a maximum of two stable cycles.

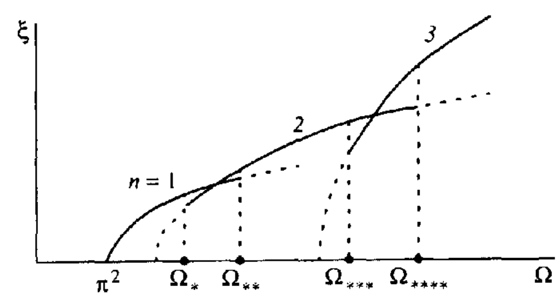

Fig. 3

2. THE BUFFERNESS PHENOMENON IN THE PROBLEM OF A STRING INTERACTING WITH AN OSCILLATOR

Formulation of the problem and description of the result. Let us consider a self-excited oscillatory system (Fig. 4), consisting of a uniform string of length l fixed at its ends, with a generator of mechanical vibrations attached to its centre. We will assume that the excited string is characterized by its density ρ, its tension T and the density of friction forces h; the generator is represented by a resonator consisting of a mass M, a spring of stiffness k and a non-linear friction element h_r. Let u_1 and u_2 denote the displacements of the parts of the string to the left and right of the point $l/2$, and v the displacement of the mass M from its equilibrium position. Then the equations of the vibrations of the string and the generator may be written as follows [12]:

$$\begin{gathered}\rho\frac{\partial^2 u_j}{\partial t^2}+h\frac{\partial u_j}{\partial t}=T\frac{\partial^2 u_j}{\partial x^2},\quad j=1,2\\ u_1|_{x=0}=u_2|_{x=l}=0,\quad u_1|_{x=l/2}=u_2|_{x=l/2}=v(t)\\ M\frac{d^2v}{dt^2}+h_r\frac{dv}{dt}+kv=T\left(\frac{\partial u_2}{\partial x}-\frac{\partial u_1}{\partial x}\right)\Bigg|_{x=l/2}\end{gathered}\tag{2.1}$$

It will be assumed that $h_r(v)=\lambda\,(v^2-1)$, $\lambda>0$, that is, we will consider a Van der Pol characteristic. Then, assuming, by symmetry, that

$$u_1(t,x)=u_2(t,l-x)=u(t,x),\quad 0\leqslant x\leqslant l/2$$

and suitably normalizing the variables t and x, we transfer from problem (2.1) to the following boundary-value problem in the interval $0\leqslant x\leqslant 1$

$$\begin{gathered}\frac{\partial^2 u}{\partial t^2}+\varepsilon\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}\\ u|_{x=0}=0,\quad\left[\frac{\partial^2 u}{\partial t^2}+\varepsilon\alpha(u^2-1)\frac{\partial u}{\partial t}+\beta u\right]\Bigg|_{x=1}=-\gamma\frac{\partial u}{\partial x}\Bigg|_{x=1}\end{gathered}\tag{2.2}$$

Throughout what follows, we will assume that $0<\varepsilon\ll 1$, and that the parameters α, β and γ are positive and of the order of unity.

As the phase space of boundary-value problem (2.2) we take

$$\left(u,\frac{\partial u}{\partial t}\right)\in \mathring{W}_2^2(0,1)\times\mathring{W}_2^1(0,1)$$

where $\mathring{W}_2^m$ $(m=1,2)$ are the Sobolev spaces of functions satisfying the first boundary condition in (2.2). It can be proved by standard means that the mixed problem corresponding to (2.2) with initial data in this phase space has a unique solution: the linear equation in (2.2) is integrated along characteristics, and the result is substituted into the boundary conditions (see, e.g. [6]).

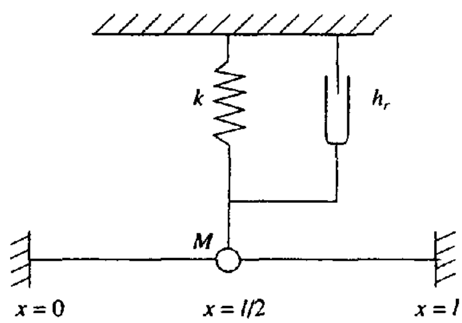

Fig. 4

Our interest lies in the existence and stability of cycles of problem (2.2) that bifurcate from zero as the parameter α is increased. We note that when $\varepsilon = 0$ the stability spectrum of the zero equilibrium state of this problem consists of pure imaginary eigenvalues $\pm i\omega_n$ ($n = 1, 2, \ldots$), where $\omega_n, n \geq 1$, are the positive roots of the equation

$$P(\omega) \equiv (\beta - \omega^2)\sin\omega + \gamma\omega\cos\omega = 0 \tag{2.3}$$

numbered in order of increasing magnitude. Corresponding to these eigenvalues are Lyapunov–Floquet solutions

$$u = \exp(\pm i\omega_n t)\sin\omega_n x \tag{2.4}$$

Thus, we will be concerned with t-periodic solutions of problem (2.2) with frequencies close to ω_n; an algorithm to determine the asymptotic behaviour of such solutions will be presented below.

Following a previously described technique [3] and taking into consideration that when $\varepsilon = 0$ problem (2.2) has periodic solutions (2.4), we fix an arbitrary n and substitute into (2.2) the series

$$u = u_0(\tau, x) + \varepsilon u_1(\tau, x) + \varepsilon^2 u_2(\tau, x) + \ldots, \quad \tau = (1 + \varepsilon\delta_1 + \varepsilon^2\delta_2 + \ldots)t \tag{2.5}$$

$$u_0(\tau, x) = \xi[\exp(i\omega_n\tau) + \exp(-i\omega_n\tau)]\sin\omega_n x \tag{2.6}$$

where ξ, δ_k, $k \geq 1$ are real constants, yet to be determined, and the functions u_k, $k \geq 1$, are odd trigonometric polynomials in $\omega_n\tau$ of degree not higher than $2k + 1$. Then, equating the coefficients of ε, we obtain the following boundary-value problem for determining u_1

$$\left(\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial x^2}\right)u_1 = -2\delta_1\frac{\partial^2 u_0}{\partial\tau^2} - \frac{\partial u_0}{\partial\tau}, \quad u_1|_{x=0} = 0$$

$$\left[\frac{\partial^2 u_1}{\partial\tau^2} + \beta u_1 + \gamma\frac{\partial u_1}{\partial x}\right]_{x=1} = -2\delta_1\left.\frac{\partial^2 u_0}{\partial\tau^2}\right|_{x=1} + \left[\alpha(1 - u_0^2)\frac{\partial u_0}{\partial\tau}\right]_{x=1}$$

The solution will be sought in the same form as the corresponding inhomogeneities, that is, in the form

$$u = A(x)\exp(i\omega_n\tau) + \bar{A}(x)\exp(-i\omega_n\tau) + B(x)\exp(3i\omega_n\tau) + \bar{B}(x)\exp(-3i\omega_n\tau)$$

Proceeding in this way, we obtain the following problems for the coefficients A and B

$$A'' + \omega_n^2 A = \omega_n\xi(i - 2\delta_1\omega_n)\sin\omega_n x$$

$$A(0) = 0, \quad (\beta - \omega_n^2)A(1) + \gamma A'(1) = [2\delta_1\omega_n + i\alpha(1 - \xi^2\sin^2\omega_n)]\xi\omega_n\sin\omega_n$$

$$B'' + 9\omega_n^2 B = 0$$

$$B(0) = 0, \quad (\beta - 9\omega_n^2)B(1) + \gamma B'(1) = -i\omega_n\alpha\xi^3\sin^3\omega_n$$

Analysis of these problems yields the equalities

$$\begin{aligned} &\xi^2 = (\alpha - \alpha_n)/(\alpha\sin^2\omega_n), \quad A(x) = -(\xi/2)ix\cos\omega_n x + \eta\sin\omega_n x \\ &\delta_1 = 0, \quad B(x) = -i\omega_n\alpha\xi^3\sin^3\omega_n\sin 3\omega_n x/P(3\omega_n) \end{aligned} \tag{2.7}$$

where

$$\alpha_n = (\omega_n^2 - \beta)^2/(2\gamma\omega_n^2) - (\omega_n^2 - \beta)/(2\omega_n^2) + \gamma/2$$

and η is an arbitrary real constant; the function $P(\omega)$ is the same as in (2.3).

Let us assume that the following conditions are satisfied

$$\alpha > \alpha_n, \quad P(3\omega_n) \neq 0 \tag{2.8}$$

Then ξ can be found from the first equality of (2.7), and consequently the function $u_1(\tau, x)$ is completely determined. In principle, this information is sufficient to prove the existence of a cycle of problem (2.2) in the zeroth approximation (2.6) and to investigate its stability. However, if necessary, this algorithm may be extended: at the mth step, where $m \geqslant 1$, the solvability of the boundary-value problem for the coefficient of u_m in the first harmonic is achieved by a term of the form $\eta \sin \omega_n x$, to within which the analogous coefficient in u_{m-1} was determined (see (2.7)), and by a correction to the frequency δ_m. Solvability of the boundary-value problems for the remaining coefficients of the function u_m is guaranteed by the inequalities

$$P(k\omega_n) \neq 0, \quad k = 3, \ldots, 2m+1$$

which characterize a certain generality of the position.

Consider the numbers

$$R_{n,m} = 2\alpha_n - \alpha_m - \alpha, \quad m = 1, 2, \ldots \tag{2.9}$$

Theorem 3. Suppose, for some natural number n, condition (2.8) holds and in addition

$$P(\omega_m \pm 2\omega_n) \neq 0, \quad m = 1, 2, \ldots, m \neq n; \quad \sin 2\omega_n \neq 0 \tag{2.10}$$

Suppose, further, that all the numbers (2.9) are non-zero and that exactly m_0 of them are positive. Then $\varepsilon_n > 0$ exists such that, for $0 < \varepsilon \leqslant \varepsilon_n$, boundary-value problem (2.2) has a periodic solution with asymptotic behaviour (2.5)–(2.7), which is exponentially orbitally stable if $m_0 = 0$, and dichotomic if $m_0 > 0$, with unstable manifold of dimension $2m_0 + 1$.

An algorithm for stability analysis. The original part of the proof of Theorem 3 consists of applying the algorithm described in [13] for stability analysis to the boundary-value problem

$$\frac{\partial^2 h}{\partial \tau^2} + \varepsilon \frac{\partial h}{\partial \tau} = \frac{\partial^2 h}{\partial x^2}, \quad h|_{x=0} = 0 \tag{2.11}$$

$$\left.\left\{\frac{\partial^2 h}{\partial \tau^2} + \varepsilon\alpha \frac{\partial}{\partial \tau}[(u_0^2(\tau, x) - 1)h] + \beta h\right\}\right|_{x=1} = -\gamma \left.\frac{\partial h}{\partial x}\right|_{x=1}$$

obtained by linearization of problem (2.2) at the approximate periodic solution constructed above, with terms of order ε^2 and higher omitted.

The gist of the algorithm is as follows. For $m \neq n$, we put

$$h = [\exp(i\omega_m \tau) \sin \omega_m x + \varepsilon h_m(\tau, x)] \exp(\varepsilon \mu_m \tau)$$

in (2.11) and compare the coefficients of ε. This yields a linear inhomogeneous boundary-value problem for h_m, while the unknown constant μ_m is determined from the condition that this problem be solvable in the class of trigonometric polynomials in $\omega_n \tau$, $\omega_m \tau$. Proceeding in this way, we obtain

$$\mu_m = R_{n,m}/(2\alpha_m), \quad h_m = C_m^0(x)\exp(i\omega_m \tau) + C_m^+(x)\exp[i(\omega_m + 2\omega_n)\tau] +$$

$$+ C_m^-(x)\exp[i(\omega_m - 2\omega_n)\tau]$$

$$C_m^0 = -(i/2)(2\mu_m + 1)x \cos \omega_m x$$

$$C_m^\pm(x) = -\alpha\xi^2 i(\omega_m \pm 2\omega_n)\sin^2 \omega_n \sin(\omega_m \pm 2\omega_n)x / P(\omega_m \pm 2\omega_n)$$

The reader's attention is drawn to the fact that the denominators in the formulae for $C_m^\pm$ are non-zero by virtue of conditions (2.10). We also note that, since

$$|P(\omega_m \pm 2\omega_n)|/\omega_m^2 \to |\sin 2\omega_n|, \quad m \to \infty$$

the role of the inequality $\sin 2\omega_n \neq 0$ also becomes clear (see (2.10)).

When $m = n$, i.e. at a natural frequency of the self-excited oscillations, our course of action is slightly different. In this case, setting

$$h = [V_0(\tau, x) + \varepsilon V_1(\tau, x)]\exp(\varepsilon D\tau), \quad D = \begin{Vmatrix} d_1 & d_2 \\ \bar{d}_2 & \bar{d}_1 \end{Vmatrix}$$

$$V_j = [v_{j,n}(\tau, x), \bar{v}_{j,n}(\tau, x)], \; j = 0, 1, \; v_{0,n} = \exp(i\omega_n \tau)\sin \omega_n x$$

in (2.11) and operating as indicated previously, we verify that

$$d_1 = d_2 = (\alpha_n - \alpha)/(2\alpha_n)$$

It is obvious from the structure of the matrix D and from the explicit form of the numbers μ_m that the formal stability properties of problem (2.11) are precisely those described in the statement of the theorem. The rigorous proof, and the proof that problem (2.2) has a cycle with the asymptotic behaviour we have constructed, are based on methods described in [14, 15], which were further developed in the monograph [6].

Conclusion. It follows from the statements of Theorem 3 that when $\alpha > \alpha_n$ a cycle with frequency close to ω_n bifurcates from the zero equilibrium position of problem (2.2); this cycle, increasing in amplitude, becomes stable when the parameter α is increased further, namely, when

$$\alpha > 2\alpha_n - \min_{m \geqslant 1} \alpha_m \tag{2.12}$$

Thus, by suitably increasing α and reducing ε, provided the position is sufficiently general with respect to the parameters β and γ, one can guarantee the existence of any previously given finite number of stable cycles of problem (2.2); that is, the *bufferness phenomenon* is observed.

We note one further tendency which is characteristic of the problem: if γ is reduced, with all other parameters held fixed, the bufferness phenomenon breaks down. Put more precisely: in that case problem (2.2) is left with a single stable cycle of frequency close to that of the oscillator. Indeed, let us assume first that $\gamma = 0$. Then problem (2.2) is equivalent to the boundary-value problem

$$\frac{\partial^2 u}{\partial t^2} + \varepsilon \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}; \quad u|_{x=0} = 0, \quad u|_{x=1} = v(t) \tag{2.13}$$

$$\frac{d^2 v}{dt^2} + \varepsilon\alpha(v^2 - 1)\frac{dv}{dt} + \beta v = 0 \tag{2.14}$$

But (2.14) is the classical Van der Pol equation, which has a single orbitally exponentially stable cycle $v = v_*(t, \varepsilon)$. Substituting this cycle into (2.13) we uniquely determine a periodic function $u = u_*(t, x, \varepsilon)$ with the same period.

When $0 < \gamma \ll 1$, the situation is in principle the same. To simplify matters, let us assume that $\beta \neq n\pi$ ($n = 1, 2, \ldots$). Then, as is easily seen, Eq. (2.3) has a simple root $\omega = \omega_0(\gamma) : \omega_0(0) = \sqrt{\beta}$, corresponding to which there is a critical value $\alpha = \alpha_0(\gamma) : \alpha_0(0) = 0$; the other roots $\omega_n(\gamma) : \omega_n(0) = n\pi$ are also associated with critical values $\alpha_n = \alpha_n(\gamma)$ of the parameter α such that $\alpha_n(\gamma) \to +\infty$ as $\gamma \to 0$. Consequently, for any fixed α and β, when $\gamma \to 0$ the existence and stability condition (2.12) for a cycle will be satisfied only at frequency ω_0.

Thus, for small γ, that is, when the reaction force exerted by the string on the oscillator is weak, one observes a *capture phenomenon*: a unique periodic regime exists with frequency close to that of the external action. On the other hand, when $\gamma \gg 1$ the vibrations in the system are completely suppressed. Thus, the bufferness phenomenon in this problem occurs in a certain intermediate range of γ values.

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REFERENCES

1. ZAKHAROV, A. A. and KOLESOV, Yu. S., Spatially inhomogeneous regimes in the predator-victim problem. In *Non-linear Oscillations in Ecology*. Izd. Yaroslav. Univ., Yaroslavl, 1984, pp. 3–15.
2. KOLESOV, A. Yu. and ROZOV, N. Kh., Diffusion bufferness in a mathematical model of biology. *Izv. Ross. Akad. Nauk, Ser. Mat.*, 1998, **62**, 5, 135–164.
3. VITT, A. A., Distributed self-excited oscillatory systems. *Zh. Tekh. Fiz.* 1934, **14**, 1, 144–157.

4. AZ'YAN Yu. M. and MIGULIN, V. V., Self-excited oscillations in a system with delayed feedback. *Radiotekhnika i Elektronika*, 1956, **1**, 4, 418–427.
5. KOLESOV, A. Yu., MISHCHENKO, Ye. F. and ROZOV, N. Kh., A parametric bufferness phenomenon in systems of parabolic and hyperbolic equations with small diffusion. *Ukr. Mat. Zh.*, 1998, **50**, 1, 22–35.
6. KOLESOV, A. Yu., MISHCHENKO, Ye. F. and ROZOV, N. Kh., Asymptotic methods of investigating periodic solutions of non-linear hyperbolic equations. *Trudy Mat. Inst. Akad. Nauk*, 1998, **222**, 3–191.
7. KAMBULOV, V. F., KOLESOV, A. Yu. and ROZOV, N. Kh., Investigation of self-excited oscillations in a Van der Pol generator with distributed parameters. *Vestnik Mosk. Gos. Univ., Ser. 3, Fizika, Astronomiya*, 1999, 3, 23–27.
8. KOLESOV, A. Yu., ROZOV, N. Kh. and SUSHKO, V. G., Specific features of self-excited oscillatory processes in resonant hyperbolic systems. *Fund. i. Prikl. Mat.*, 1999, **5**, 2, 437–473.
9. BOLOTIN, V. V., *Non-conservative Problems in the Theory of Elastic Stability*. Fizmatgiz, Moscow, 1961.
10. SOBOLEVSKII, P. Ye., On equations of parabolic type in Banach space. *Trudy Mosk. Mat. Obshch.*, 1961, **10**, 297–350.
11. HENRY, D., *Geometric Theory of Semilinear Parabolic Equations*. Springer, Berlin, 1981.
12. MIGULIN, V. V., MEDVEDEV, V. I., MUSTEL', Ye. R. and PARYGIN, V. N., *Principles of Oscillation Theory*. Nauka, Moscow, 1988.
13. KOLESOV, Yu. S. and MAIOROV, V. V., A new method for investigating the stability of solutions of linear differential equations with almost constant almost periodic coefficients. *Differents. Uravneniya*, 1974, **10**, 10, 1778–1788.
14. KOLESOV, Yu. S., Bifurcation of invariant tori of parabolic systems with small diffusion. *Mat. Sbornik*, 1993, **184**, 3, 121–136.
15. KOLESOV, Yu. S., The asymptotic form and stability of non-linear parametric oscillations of a singularly perturbed telegraph equation. *Mat. Sbornik*, 1995, **186**, 10, 57–72.

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